WWW.CALCULUS-HELP.COM'S

2004

Superbowl of High School Calculus

Final Reminders and Last Requests:

1. Members of the same group may work together however they wish, but separate groups may not work together.
2. You may not use a calculator, computer, abacus, pile of smooth stones, or any other computing device of any type or level of technology.
3. Show all your work for every step of every problem, and be as neat as possible. Organization and presentation may serve as a tie-breaker to comparable solutions! Attach additional sheets as necessary.
4. Please simplify all answers as much as possible; that means you need to evaluate any trigonometric functions on the unit circle. (For example, don't leave "$\cos\pi$" as a solution. The answer would be "–1.")
5. Unless explicitly stated otherwise, assume all angles are expressed as radians.
6. If one part of a problem refers to a function, like $f(x)$, you can assume that any other references to $f(x)$ in the other parts of the same problem refer to the same function. However, if you see that notation in a completely different problem, it probably refers to a completely different function.
7. After exactly 3 hours, it's all over but the crying (unless you have an authorized extension).
8. The test administrator (teacher or staff member) should sign the bottom of this form when the test is over to indicate that the participants worked only in groups of 3, got no additional help, and did not use restricted technology, books, or notes.
9. **Submit only one solution per problem per team!**

Team Name: ____________________ School Name: ______________________

Team Members: ____

By signing below, I pledge that this team consisted of only the 3 members listed above; they did not use technology, books, or notes to answer questions; and they did not benefit from help from anyone outside the group including (but not limited to) other teams, teachers, school officials, parents, and the disembodied spirit of Troubled Rocker Kurt Cobain.

Test administrator's signature: ____

All the Legal Stuff

Note that the rubric attached to each question represents only a *guideline* for grading; the judge's word is final on all grades, even if you disagree. Remember that all answers were required to be fully simplified, and that (unlike the AP test) correct notation counts! Also unlike AP test, answers that depend on and incorporate incorrect earlier parts of the question may not receive any credit.

Question 1: Mystery Date (The Function Edition)

Some function $f(x)$ has each of the following characteristics:

- Exactly two roots: $x=1$ and $x=-\frac{3}{2}$
- Exactly two vertical asymptotes, with equations $x=7$ and $x=-\frac{1}{2}$
- $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 1$
- Though both are defined, $\lim_{x\to-3} f(x) \neq f(-3)$

(a) Write the function $f(x)$ in simplest form.

(b) Evaluate $f'(1)$.

Rubric

(a) **1 point each:** $(x-1),(2x+3)$ in numerator; $(x-7),(2x+1)$ in denominator, total of 4 points available

3 points: Appropriately handle $\lim_{x\to-3} f(x) \neq f(-3)$ condition, but cannot contradict earlier parts of problem

Most common solution receiving full credit:

$$f(x) = \begin{cases} \dfrac{2x^3+7x^2-9}{2x^3-7x^2-46x-21}, & x \neq -3 \\ 0, & x=-3 \end{cases}$$

(b) **3 points:** $-\frac{5}{18}$ for preceding common answer, or answer that works for fully justifiable alternative solution

Question 2: Does the Water Get Him Instead? Nobody Knows.

The velocity of a particle moving along the *x*-axis at any time $t > 0$ is modeled by the function $v(t) = (\sin 3t)e^{\cos 3t}$.

(a) How many times, if any, does the particle change direction from time $t = 0$ to time $t = 2\pi$?

(b) Describe the speed and acceleration of the particle when $t = \frac{\pi}{3}$.

(c) If the particle is positioned e units to the right of the origin when $t = 0$, what function, $s(t)$, correctly models the position of the particle when $t > 0$?

(d) Write, but do not evaluate, the expression representing the total distance traveled by the particle between $t = 0$ and $t = 2\pi$.

Rubric

(a)	**2 points:**	5
(b)	**1 point:**	speed = 0
	2 points:	acceleration = $-\frac{3}{e}$
(c)	**3 points:**	$s(t) = -\frac{1}{3}e^{\cos 3t} + \frac{4}{3}e$
(d)	**2 points:**	$\int_0^{2\pi} \lvert v(t) \rvert \, dt$

Question 3: Back to Fundamentals

Given the function $g(x)=\int_{2a}^{0}\left(t^{b}-\sin ct\right)dt$, assume that *a*, *b*, and *c* are real numbers.

(a) Evaluate $g\left(\frac{\pi}{2}\right)$.

(b) Evaluate $g'\left(\frac{\pi}{2}\right)$.

(c) Describe how you'd calculate the average rate of change of $g'(x)$ on a closed interval over which $g'(x)$ is antidifferentiable, without calculating $g''(x)$.

Rubric

***Note:** This problem was specifically designed to "look like" a Fundamental Theorem of Calculus problem for which you could use the $\frac{d}{dx}\left(\int_{a}^{f(x)}g(t)dt\right)=g(f(x))f'(x)$ shortcut*, but neither of the limits of integration contains *x*, the variable in which *g* is said to be defined. The discerning calculus student must, then, draw the conclusion that it is a constant function, and parts (b) and (c) become trivial. Part (c) advises against calculating the average rate of change, but this is a red herring; the correct answer of 0 is expected, because no calculation is truly necessary. Good arguments and explanations (good, not *long*) accompanied by correct answers earn full credit. Moral of the story: Even if you think there's a typo on a test (which is not true for this problem), solve it as written. Don't get clever and invent your own problem, because it won't match my answer.

(a) **4 points:** $-\frac{2a^{b+1}}{b+1}-\frac{\cos(2ac)-1}{c}$

(b) **4 points:** Answer of 0 with justification, such as "*g*(*x*) is a constant function, and constant functions always have a derivative of 0"

(c) **2 points:** 0: the rate of change of the constant 0 is 0

Question 4: Chart Art

Two functions, $f(x)$ and $g(x)$, and their derivatives, $f'(x)$ and $g'(x)$, are continuous and differentiable for all real numbers. Some values for the functions are given in the table below.

x	–1	0	1	2	3
$f(x)$	2	1	4	–3	–1
$f'(x)$	0	2	–2	$\frac{1}{2}$	6
$g(x)$	3	–1	2	1	–2
$g'(x)$	–5	–2	0	1	7

(a) Evaluate $(f+g)'(2)$.

(b) Evaluate $(fg)'(-1)$.

(c) Evaluate $\left(g^{-1}\right)'(-1)$.

(d) Approximate $g''(4)$ and show the work that leads to your solution.

Rubric

(a) **2 points:** $\frac{3}{2}$

(b) **3 points:** –10 (more than half of students did not apply product rule)

(c) **3 points:** $-\frac{1}{2}$

(d) **2 points:** Justifiable answer with explanation; most common solution used secant line connecting $g'(2)$ and $g'(3)$ to get $g''(4)=6$

Question 5: Related Rates in the Membrane, Insane in the Range

The below items refer to the equation $3y^2 - y = x^2 + 2x - 4$.

(a) Calculate the instantaneous rate of change at the generic point (a,b) on the curve.

(b) Describe the relationship defining the set of ordered pair (m,n) on the curve whose *x*- and *y*-coordinates are changing at the same rate.

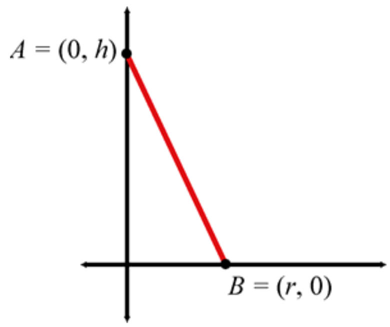
Rubric

(a) **5 points:** $\frac{dy}{dx} = \frac{2a+2}{6b-1}$

(b) **5 points:** $2m - 6n = -3$; though you can use part (a) to reach this answer, full credit was not given for doing so without a substantial and mathematically sound explanation—better just to begin again and show sound steps than start in the middle and try to muddle your way through and get the right answer without knowing why

Question 6: Coney Island

(a) Generate the formula for the volume of a right circular cone by rotating the region bounded by the *x*- and *y*-axes and line segment $\overline{AB}$ (pictured below) about the *y*-axis.



(b) Assume that the region described in part (a) is the base of a solid whose cross-sections are equilateral triangles perpendicular to the *x*-axis. Find the volume of the solid.

Rubric

(a) **2 points:** Disc method setup $\pi \cdot \int_0^h \left[-\frac{r}{h}(y-h)\right]^2 dy$ or equivalent shell method setup

3 points: Correct steps leading to answer of $\frac{1}{3}\pi r^2 h$; simply knowing and writing the cylinder volume formula gets no points on its own

(b) **2 points:** Determining (by memory or geometry) that the area of an equilateral triangle with side *s* equals $\frac{\sqrt{3}}{4}s^2$

3 points: Work leading to correct answer of $\frac{\sqrt{3}}{12}h^2 r$

Question 7: Log Jam

Assume $h(x)$ is the piecewise-defined function below:

$$h(x)=\begin{cases} x^2-kx-21 & , x\le 8 \\ \log_2 x & , x>8 \end{cases}$$

(a) Find the value of k which makes $h(x)$ continuous for all real numbers.

(b) The function $f(x)$ is defined as $h(x)$ with the correct value of k (from part (a) above) substituted in. For what value(s) of x does $f(x)$ satisfy the Mean Value Theorem on the interval $[0,16]$?
Note: You must get part (a) correct in order to earn full credit for part (b).

(c) Describe in detail the process you'd use to evaluate the expression $\int_1^{10} h(x)\,dx$, indicating any additional information required to reach a solution, if any is required.

Rubric

(a) **3 points:** 5

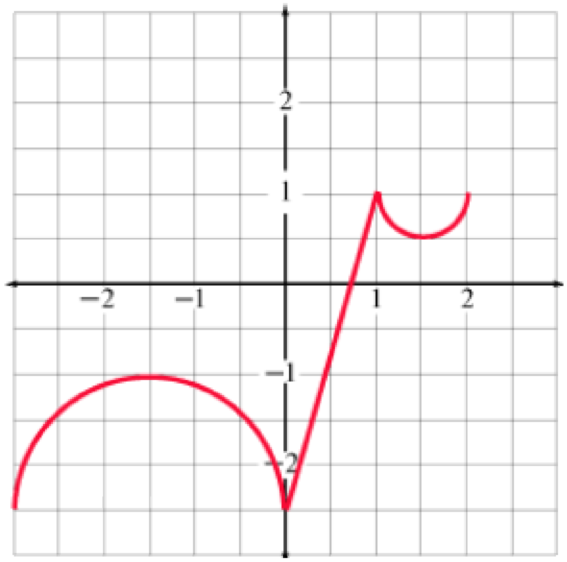
(b) **4 points:** $\frac{105}{32}$

(c) **2 points:** $\int_1^8 \left(x^2-5x-21\right)dx+\int_8^{10}\log_2 x\,dx$

1 point: You'd need to know how to integrate the logarithmic function, which is not part of the AB curriculum; more advanced students could get credit by identifying integration by parts, as nearly all of them did

Question 8: Humps and Bumps That Kick Your Rumps

A function $g(x)$ is continuous on the closed interval $[-3,2]$, and $g(0)=1$. The graph of its derivative, $g'(x)$, is comprised of the line segment and semicircles pictured below.



(a) On what interval(s) is $g(x)$ increasing?

(b) Write the equations of the tangent lines to $g(x)$ and $g'(x)$ when $x=0$.

(c) For what value(s) of x is the graph of $g(x)$ **NOT** concave up?

(d) Does $g(1)$ possess a real number value? Why or why not?

(e) Evaluate $g(-3)$ and $g(2)$.

Rubric

(a) **2 points:** $\left(\frac{5}{7},2\right)$**;** endpoint inclusion/exclusion irrelevant

(b) **2 points:** $5x + 2y = 2$, $g(x)$; $g'(x)$ has no tangent line there, $g''(0)$ doesn't exist

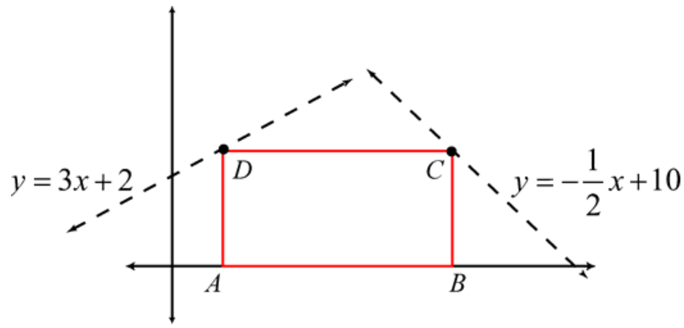
(c) **2 points:** $\left(-\frac{3}{2},0\right)\cup\left(1,\frac{3}{2}\right)$

(d) **2 points:** Yes, $g(x)$ is described as continuous on $[-3,2]$.

(e) **2 points:** $g(-3)=\dfrac{68-9\pi}{8}$; $g(2)=\dfrac{10-\pi}{8}$

Question 9: Laying It on The Line

Points A and B of rectangle $ABCD$ lie on the positive x-axis, as shown in the figure below. Point C lies on the line $y=-\frac{1}{2}x+10$, and D lies on the line $y=3x+2$. Find the maximum area of $ABCD$.



Note: Picture may not be drawn to scale.

Pseudo-Rubric

10 points: Work leading to correct answer of $\frac{961}{21}$; note that the question asks for the maximum area, not the maximum dimensions

Question 10: Enter the Denominatrix

Assume that $j(x)=\dfrac{a}{bx+c}$ is a rational function in simplest form containing real numbers a, b, and c, such that $a<0<b<c$.

(a) If $m=-\dfrac{c}{b}$, evaluate $\lim\limits_{x\to m^+} j(x)$ and justify your answer.

(b) If p is a real number in the domain of $j(x)$ and the domain of $j^{(n)}(x)$, where n is a positive integer, calculate $j''(p)$.

(c) Assume $j(x)$ is continuous on the interval $[h,k]$, whose boundaries are real numbers such that $h<k$. What is the average value of $j(x)$ on $[h,k]$?

Rubric

(a) **1 point:** $-\infty$

2 points: Explanation of solution, including mention of discontinuity or asymptotic behavior and proper justification of why answer is not, therefore, ∞.

(b) **3 points:** $\dfrac{2ab^2}{(bp+c)^3}$

(c) **2 points:** $\dfrac{1}{k-h}\displaystyle\int_h^k \frac{a}{bx+c}\,dx$

2 points: $\dfrac{a}{b(k-h)}\ln\left(\dfrac{bk+c}{bh+c}\right)$

Statistical Analysis for 2004 Superbowl Questions

Question	Mean	Median	Mode	Standard Deviation
1. Mystery Date	5.628	7	7	3.456
2. Does the Water Get Him Instead?	5.489	6	10	3.234
3. Back to Fundamentals	3.511	2	0	3.498
4. Chart Art	4.361	4	2	2.737
5. Related Rates in the Membrane	6.222	5	5	2.960
6. Coney Island	4.367	3.25	0	3.728
7. Log Jam	4.933	5	0	3.376
8. Humps and Bumps	4.428	4.75	6	2.797
9. Laying It on The Line	3.228	0	0	4.283
10. Enter the Denominatrix	4.061	3	0	3.838
Total Score	**46.228**	**42.75**	**25**	**24.470**